



Note

A note on a problem of Erdős on right angles[☆]Gy. Elekes¹

ARTICLE INFO

Article history:

Received 25 February 2009

Accepted 28 February 2009

Available online 11 April 2009

Keywords:

Plane

Points

Angle

ABSTRACT

Erdős asked the following question: is it true that a set of n^2 points in the plane always contain $2n - 2$ points which do not determine a right angle? We show that, apart from a log-factor, the answer is in the affirmative.

© 2009 Published by Elsevier B.V.

In [1] Paul Erdős asked the following question: is it true that a set of n^2 points in the plane always contain $2n - 2$ points which do not determine a right angle? An $n \times n$ square lattice shows that this bound, if true, is best possible. Erdős could prove the existence of a subset of $cn^{2/3}$ such points.

We show that, apart from a log-factor, the answer is in the affirmative — even for any other given angle β in place of 90° .

Theorem 1. *Let $\beta \in (0, \pi)$ be an arbitrary angle and \mathcal{P} a set of N points in the Euclidean plane. Then it is possible to select a subset $\mathcal{P}_0 \subset \mathcal{P}$ of size at least*

$$|\mathcal{P}_0| \geq c \cdot \frac{\sqrt{N}}{\sqrt{\log N}}$$

such that no three points of \mathcal{P}_0 determine an angle β . Here c is an absolute constant, independent of N .

The proof uses two tools. The first one is a bound of Pach and Sharir [4].

Theorem 2 ([4]). *In any set of N points in the Euclidean plane and for any $\beta \in (0, \pi)$, at most $c_1 N^2 \log N$ triples determine the angle β , where c_1 is an absolute constant. \square*

The other tool we need is a lower bound on the number of elements of a triple system \mathcal{T} that does not contain any triples — it is called the independence number of a triple system and denoted by $\alpha(\mathcal{T})$. Such a bound is provided by Spencer's result [5] on the independence number of uniform hypergraphs (see also in [3], p. 150).

Theorem 3 ([5]). *If a triple system \mathcal{T} (a 3-uniform hypergraph) has m triples on N elements then*

$$\alpha(\mathcal{T}) \geq c_2 \sqrt{\frac{N^3}{m}}$$

with a constant c_2 .

[☆] Please send all correspondence to András Gyárfás at gyarfasa@gmail.com.

¹ Deceased author.

Theorem 1 follows by substituting the bound of **Theorem 2** to **Theorem 3**.

The proof method can be applied to other properties of triples of points. For example, in [4] it was shown that N points determine at most $O(N^{\frac{7}{3}})$ triangles with any of the following properties: isosceles; having the same area; having the same perimeter. By the argument above it follows that from N points one can always select at least $O(N^{\frac{1}{3}})$ such that among the selected points no triangle has the property (being isosceles, having given area or perimeter).

Addendum

Due to the untimely death of the author, the acknowledged added some sentences to the manuscript. Pach pointed out reference [5] to **Theorem 3** (it was reproved by Elekes in an earlier version of this manuscript).

Mubayi noted that – if the following version of his recent conjecture with Frieze [2] is true – $\sqrt{\log N}$ could be removed from the denominator of **Theorem 1** and thus the right order of magnitude could be reached in the Erdős conjecture.

Conjecture. *If a triple system \mathcal{T} (a 3-uniform hypergraph) has a fixed forbidden subhypergraph and has average degree d on N elements then*

$$\alpha(\mathcal{T}) \geq c_3 N \sqrt{\frac{\log d}{d}}$$

with a constant c_3 .

The following lemma shows that the hypergraph K_9^3 (all triples in nine elements) is forbidden from the triples that determine angle β .

Lemma. *Assume that $0 < \beta < \pi$ and all triangles determined by m points of the plane have an angle β . Then $m \leq 8$.*

Proof. Let S be a set of $m \geq 9$ points, from the condition, no three points of S can be collinear. This implies that around any point $p \in S$ the other $m - 1$ points can be ordered clockwise, thus each x has at most one clockwise neighbor y with $\angle xpy = \beta$. Therefore $\angle xpy = \beta$ holds for at most $m - 1$ pairs $x, y \in S$. However, some $p \in S$ is in at least $\frac{\binom{m}{3}}{m} > m - 1$ angles $\angle xpy$ – a contradiction if $m \geq 9$. \square

By **Theorem 2** the average degree of the hypergraph of triples with an angle β is at most $O(N \log N)$. Therefore, from the Conjecture, $\alpha(\mathcal{T}) \geq c_4 \sqrt{N}$ would follow, implying the claimed improvement of **Theorem 1**.

Acknowledgment

We are grateful to András Gyárfás for useful conversations on the topic.

References

- [1] P. Erdős, Problems and results in combinatorial analysis, in: Proceedings of the 8-th Southeastern Conference on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, La., 1977), in: Congress. Numer., vol. XIX, Utilitas Math., Winnipeg, 1977, pp. 3–12.
- [2] A. Frieze, D. Mubayi, Coloring simple hypergraphs, manuscript. <http://www.math.uic.edu/~mubayi/papers/ksimple.pdf>, 2008.
- [3] J. Pach, P.K. Agarwal, Combinatorial Geometry, Wiley-Interscience, 1995.
- [4] J. Pach, M. Sharir, Repeated angles in the plane and related problems, Journal of Combinatorial Theory A 59 (1990) 12–22.
- [5] J. Spencer, Turán's theorem for k -graphs, Discrete Mathematics 2 (1972) 183–186.